Čech Cohomology

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§. Čech Complex

1∘ <u>Definition</u> n-simplex on a set I

let I be a set & $[0, n] = \{ n \in \mathbb{N} ; 0 \le m \le n \}$

An <u>n-simplex</u> is a function $\sigma : [0, n] \to I$

& let I_n be set of n-simplices.

S.t. for $0 < m \le n+1$; $\exists$ maps

(*) $\partial_m : I_{n+1} \longrightarrow I_n$ (omit m^{th} vertex)

$$\sigma \longmapsto \left(\sigma' : k \longmapsto \begin{cases} \sigma(k) & \text{if } k < m \\ \sigma(k+1) & \text{if } k \ge m \end{cases} \right)$$

Čech Complex

2∘ let X be a top. space & $\mathcal{F}$ be PSh$/X$ (of abelian gps)

set $\mathcal{U} := (U_i)_{i \in I}$ open cover of X.

For n-simplex $\sigma \in I_n$; let $U_\sigma := \bigcap \{ U_{\sigma(m)} ; m \in [0, n] \}$

(intersection of members of $\mathcal{U}$ & so open set of X)

For $n \in \mathbb{N}$, $C^n(\mathcal{U}, \mathcal{F}) = \prod_{\sigma \in I_n} \mathcal{F}(U_\sigma)$

The ∂_m (*) induces

$$\partial_m : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F})$$

$$\overset{\in}{(s_\sigma)} \longmapsto \overset{\in}{(t_{\sigma'})} = s_{\partial_m \circ \sigma'}$$

& set $\forall n$:

$$d_n = \sum_{m=0}^{n+1} (-1)^m \partial_m : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F}) \quad \text{(coboundary)}$$

(Apply $d \circ d \Rightarrow$ each term appears twice w opposite signs)

Thus $d^2 = 0$

So $C^\bullet(\mathcal{U}, \mathcal{F})$ becomes a complex : Čech Complex
(belonging to $\mathcal{U}, \mathcal{F}$)

• w/ Cohomology : $\check{H}^n(\mathcal{U}, \mathcal{F}) = \check{H}^n(C^\bullet(\mathcal{U}, \mathcal{F}))$

3° <u>Example</u>. $\check{H}^0(\mathcal{U},\mathcal{F})$ is kernel of the map.

$$\begin{array}{ccc} C^0(\mathcal{U},\mathcal{F}) & \longrightarrow & C^1(\mathcal{U},\mathcal{F}) \\ \| & & \| \\ \prod_{i\in I} \mathcal{F}(U_i) & \longrightarrow & \prod_{(i,j)\in I\times I} F(U_i\cap U_j) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (s_i) & \longmapsto & \left(\rho_{U_j,U_i\cap U_j}(s_j)-\rho_{U_i,U_i\cap U_j}(s_i)\right) \end{array}$$

& if $\mathcal{F}$ is a sheaf, we have $\check{H}^0(\mathcal{U},\mathcal{F})=\mathcal{F}(X)$.

§ • <u>Refinements of coverings</u>

Two open coverings $\mathcal{U}=(U_i)_{i\in I}$ & $\mathcal{V}=(V_j)_{j\in J}$

$\mathcal{V}$ <u>refinement of $\mathcal{U}$</u> iff $\exists$ refinement $r: J\to I$ s.t.

$$\forall j\in J \qquad V_j\subseteq U_{r(j)}$$

- This induces a morphism of complexes

$$r^\bullet: C^\bullet(\mathcal{U},\mathcal{F})\to C^\bullet(\mathcal{V},\mathcal{F})$$

from $\prod_{\sigma\in I_n}\mathcal{F}(U_\sigma)\longrightarrow\prod_{\tau\in J_n}\mathcal{F}(V_\tau)$

$$s_\sigma \longmapsto t_\tau \quad \left(= \rho^{U_{r(\tau)}}_{V_\tau}(s_{r(\tau)})\right)$$

$r(\tau)$ map induced by composition from J_n to I_n.

- $\&$ r' induces morphism of Čech cohomology

$$\bar{r} : H^n(\mathcal{U}, \mathcal{F}) \to H^n(\mathcal{V}, \mathcal{F})$$

5. Lemma If $\mathcal{V}$ is a refinement of $\mathcal{U}$ & $r_1, r_2 : J \to I$ are two refinement maps, then they induce same morphism of Čech cohomology

$$\bar{r}_1 = \bar{r}_2 : H^n(\mathcal{U}, \mathcal{F}) \to H^n(\mathcal{V}, \mathcal{F})$$

Proof. (idea: homotopic maps induce same cohomology morphism)

Maps r_1' & r_2' are homotopic morphisms of complexes by homotopy:

$$k_n : C^{n+1}(\mathcal{U}, \mathcal{F}) \to C^n(\mathcal{V}, \mathcal{F})$$
$$s_\sigma \longmapsto t_\tau$$

where $t_\tau = \sum_{k=0}^{n} (-1)^k \rho^{U_{\tau_k}}_{V_\tau}(s_{\tau_k})$

and where $\tau_k(m) = \begin{cases} r_1(\tau(m)) & \text{if } m \leq k \\ r_2(\tau(m-1)) & \text{if } m > k \end{cases}$ for $m \in [0, n+1]$

Apply 2.2 : i.e. We have homotopic maps k_n

use: Chain homotopy

$$\begin{tikzcd} & C^n(\mathcal{U}, \mathcal{F}) \arrow[r, "d_n"] \arrow[dl, "k_{n-1}"'] \arrow[d, shift left] \arrow[d, shift right] & C^{n+1}(\mathcal{U}, \mathcal{F}) \arrow[dl, "k_n"] \\ C^{n-1}(\mathcal{V}, \mathcal{F}) \arrow[r, "d'_{n-1}"] & C^n(\mathcal{V}, \mathcal{F}) & \end{tikzcd}$$

st $\forall n$, $d'_{n-1} k_{n-1} + k_n d_n = g_n - h_n$

which induce same cohomology morphism. $\square$

§. Čech cohomology of presheaf $\mathcal{F}$ on X

6. Definition

Abelian gps $H^n(\mathcal{U}, \mathcal{F})$ form a directed system as $\mathcal{U}$ varies over open covers of X. (over finer & finer covers)

$$\check{H}^n(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^n(\mathcal{U}, \mathcal{F})$$

- May happen this class of open cover of X is not a set.

  In that case

- If $\mathcal{V}$ is a refinement of $\mathcal{U}$, the morphism

$$r' : C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{V}, \mathcal{F})$$

depends on choice of r ($: J \to I$; although not $\bar{r} : H(\mathcal{U}) \to H(\mathcal{V})$)

- So may have a problem in defining direct system of $C^\bullet(\mathcal{U}, \mathcal{F})$ to be able to obtain exact cohomology sequence.

- Solution (Godement)

  let $R(X)$ be set of open covers $(U_x)_{x \in X}$ indexed by X st. $\forall x \in X$, $x \in U_x$.

- Then, define a preorder on X by

$$\mathcal{V} \geq \mathcal{U} \quad \text{iff} \quad \forall x \in X \quad V_x \subseteq U_x$$

- so we obtain a canonical refinement map $X \xrightarrow{id} X$ when $\mathcal{V} \geq \mathcal{U}$.

- This allows a morphism of complexes

$$C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{V}, \mathcal{F})$$

- Can now define $C^\bullet(X, \mathcal{F}) = \varinjlim_{\mathcal{U} \in R(X)} C^\bullet(\mathcal{U}, \mathcal{F})$

- Use fact that $\varinjlim$ is exact, then can define

$$\check{H}^n(X,\mathcal{F}) = H^n(C C^\bullet(X,\mathcal{F}))$$

§. Čech cohomology an exact ∂-functor (on PSh/X)

8. Theorem.

$\{\check{H}^n(X,-)\,;\,n\in\mathbb{N}\}$ is an exact ∂-functor

$$PSh/X \to Abgp$$

Proof. An exact seq. $0\to P\to Q\to R\to 0$ in PSh/X

$\forall\, \mathcal{U}\in \mathcal{R}(X)$ gives exact seq of complexes

$$0\to C^\bullet(\mathcal{U},P)\to C^\bullet(\mathcal{U},Q)\to C^\bullet(\mathcal{U},R)\to 0$$

Apply $\varinjlim_{\mathcal{U}\in\mathcal{R}(X)}$ & get

$$0\to C^\bullet(X,P)\to C^\bullet(X,Q)\to C^\bullet(X,R)\to 0$$

(by 2-12: since PSh/X is Ab Cat,

$\exists\,\partial: H^n(X,R)\to H^{n+1}(X,P)$ st. $\exists$ LES of cohomology)

& this means we have the desired LES of cohomology

$$0\to H^0(X,P)\to H^0(X,Q)\to H^0(X,R)\xrightarrow{\partial} H^1(X,P)\to\cdots$$

$$\to\cdots H^n(X,Q)\xrightarrow{\partial} H^{n+1}(X,P)\cdots\quad\square$$

§. Comparision to Sheaf Cohomology

To identify $\check{H}^n(\mathcal{U}, \mathcal{F}) = H^n(X, \mathcal{F})$, need to verify that resolution of Čech complex is Γ-acyclic

where $\check{H}^n : PSh/X \to Ab$

Proceed in number of steps

1. $\check{H}^0(X, P) = \Gamma(X, P) = H^0(X, P)$ (•11)

2. $\check{H}^n(X, E) = 0$ for E injective (effaceable) (•12)

3. $\check{H}^n(X, -)$ forms an exact ∂-functor if (•13)

   a. sheafification is isom., (this statement is equivalent to)

   b. An $S \in PSh/X$ having zero sheafification in which case,

$$\check{H}^n(X, S) = 0$$ (•14)

(3.) Holds for any top. space X for $n = \{0, 1\}$ (•15)
(w/ Čech-to-derived functor spectral sequence)
And if X is paracompact for any n. (•16)

In paracompact X, can verify P is Γ-acyclic (next lecture). Then

Leray's Theorem: If P sheaf on X, & $\mathcal{U}$ open cover of X.

If P is acyclic on every finite intersection of elements of $\mathcal{U}$, then

$$\check{H}^n(\mathcal{U}, P) = H^n(X, P).$$

11. By (·3) if $P \in PSh/X$ is a sheaf on X then

$$\check{H}^0(X,P) = \Gamma(X,P) = H^0(X,P)$$

If we restrict $\check{H}^n(X,-)$ to Sh/X (of ab gps) this may not be ∂-functor, because exact seq in Sh/X are not necessarily exact in PSh/X.

- A SES in Sh/X:

$$0 \xrightarrow{f} P \to Q \to R \to 0$$

has $\longrightarrow$ $R = Sh\,Coker(f)$

- Sheafifying in PSh/X we get

$$\underbrace{0 \to P \to Q \to R' = PSh\,Coker(f) \to 0}_{\text{exact in } PSh}$$

with $Q \to R$ and $R' \uparrow R$ (sheafification).

Sheafification induces natural maps

$$\Gamma(X,R) = \check{H}^0(X,R)$$

& also $\forall i > 0$
$\check{H}^i(X,R') \to H^i(X,R)$

$$\underbrace{0 \to \check{H}^0(X,P) \to \check{H}^0(X,Q) \to \check{H}^0(X,R') \to \check{H}^1(X,P) \cdots \longrightarrow}_{\text{with exact bottom row.}}$$

with $\check{H}^0(X,R') \uparrow \check{H}^0(X,R)$.

Now we are ready to compare two cohomologies :

Recall by § 5.2.13 $R^\circ\Gamma \to \check{H}^\circ$ are isomorphisms of ∂-functors if $\check{H}^\circ$ is effaceable & exact (the universal property)

12 ∘ Lemma. If E injective sheaf (of ab grps) on X for $n>0$

then $\check{H}^n(X, E) = 0$

(i.e. $\check{H}^*(X, -)$ is effaceable on Sh/X)

proof (5.4.12 Tennison) □

Now we are left to study the exactness of $\check{H}^\circ$

& if $\check{H}^\circ$ forms an exact ∂-functor Then

by UP $H^n(X,-) \cong \check{H}^n(X, -)$ for $0 \leq n \leq a$

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13 ∘ $\check{H}^\circ(X, -)$ forms an exact ∂-functor when

a. If $R' \in PSh/X$ with sheafification R Then induced Čech cohomology map $\check{H}^n(X, R') \xrightarrow{\sim} \check{H}^n(X, R)$ is an isomorphism.

$\hookrightarrow$ let $S \in PSh/X$ st. $\exists$ SES $0 \to R' \to R \to S \to 0$ in PSh/X. This means S has no sheafification (i.e. sheafification is zero sheaf) & so $\exists$ LES in cohom.

$$\check{H}^{n-1}(X,S) \to \check{H}^{n}(X,R') \to \check{H}^{n}(X,R) \to \check{H}^{n}(X,S) \to \cdots$$

Clearly (a) equivalent to saying

b. If $S \in PSh/X$ with zero sheafification, then $\check{H}^{n}(X,S) = 0$

(13.) is true for $n = \{0,1\}$ without any other hypothesis on X.

14. Theorem For $n=0$

let X be a top. space & S is a presheaf w zero sheafification then $\check{H}^{0}(X,S) = 0$

Hence, $\check{H}^{0}$ forms an exact, effaceable ∂-functor on Shv/X.

& so for any sheaf $\mathcal{F}$ on X

$$\check{H}^{0}(X,\mathcal{F}) \cong H^{0}(X,\mathcal{F})$$

Proof. 5.4.14 Tennison $\square$

- Aside: Čech-to-derived functor Spectral Sequences

Computes derived functors of composition of two functors based on information of derived functors $R^\bullet\Gamma$ & $\check{H}$.

Theorem [Grothendieck 3.8.1]

Let X be a top space $\mathcal{U}$ the open cover. Then there exists a cohomological spectral functor on Sh/X converging to graded functor $\{H^n(X,\mathcal{F})\}$ whose initial term is

$$H^p(\mathcal{U}, \mathcal{H}^q(\mathcal{F})) =: E_2^{p,q}$$

where $\mathcal{H}^q(\mathcal{F})$ denotes presheaf $U \mapsto H^q(U,\mathcal{F})$ corresponding to $U \overset{open}{\subset} X$ & $H^q(U,\mathcal{F})$ its sheaf cohomology.

This spectral sequence gives natural transformation

$$H^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F})$$

Corollary [G 3.8.2: Čech-to-derived Spectral Seq]

X arbitrary top. space. Then $\exists$ a spectral functor on Sh/X converging to $H^n(X,\mathcal{F})$ whose initial term is given by

$$E_2^{p,q}(\mathcal{F}) = \check{H}^p(X, \mathcal{H}^q(\mathcal{F}))$$

$\mathcal{H}^q(\mathcal{F})$ as above.

And the spectral sequence gives a natural transformation $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$

There exists 5-term exact sequence of low degree in S.S.

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2$$

which is

$$0 \to \check{H}^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \check{H}^0(X, \mathcal{H}^1(\mathcal{F})) \to \check{H}^2(X, \mathcal{F}) \to H^2(X, \mathcal{F})$$

ISo for $n = 1$: Need to show $E_2^{0,1} = 0$

i.e. $\check{H}^0(X, \mathcal{H}^1(\mathcal{F})) = 0$ so we have the isom.

$$\check{H}^1(X, \mathcal{F}) \cong H^1(X, \mathcal{F})$$

Proof: Use

lemma For any $\mathcal{F} \in Sh/X$, $\check{H}^0(X, \mathcal{H}^q(\mathcal{F})) = 0 \quad \forall q > 0$.

proof. We show sheafification of $\mathcal{H}^q(\mathcal{F}) = 0 \quad \forall p > 0$

Consider factorization of identical functor id_S of Sh/X:

$Sh/X \xrightarrow{ind.} PSh/X \xrightarrow{\#:\text{ sheafification}} Sh/X$. We know sheafification functor $\#$ is exact, so $R^q \# = 0$ for $q > 0$ & in particular each object in PSh/X is $\#$-acyclic. So $\exists$ $\forall \mathcal{F} \in S$ a spectral sequence $E_2^{pq} = R^p \# (\mathcal{H}^q(\mathcal{F})) \Rightarrow E^{p+q} = R^{p+q} id_S(\mathcal{F})$ which is functorial in $\mathcal{F}$.

Since $R^p \# = 0$ also for same reason as earlier we have $E_2^{pq} = 0$ for $p > 0$.

This means $R^q \, id_S(\mathcal{F}) \cong \mathcal{H}^q(\mathcal{F})^\#$ $\forall q$.

But we have $R^q \, id_S = 0$ & so $\mathcal{H}^q(\mathcal{F})^\# = 0$ as well. $\square$

Now we have $\check{H}^0(X, \mathcal{H}^q(\mathcal{F})) = 0$, in particular shown $\mathcal{H}^q(\mathcal{F})$ has zero sheafification & so with the help of the spectral sequence can identify

$$0 \to \check{H}^1(X, \mathcal{F}) \xrightarrow{\sim} H^1(X, \mathcal{F}) \to 0 \to \cdots$$

Next if $\check{H}^p$ has zero sheafication $\forall p > 0$, it will be an exact, effaceable δ-functor & we can identify it w H^p

• 16 Isomorphism of Čech & Sheaf Cohomology

<u>Require:</u> Acyclicity of sheaf on every finite intersection of elements of $\mathcal{U}$

i.e. $\mathcal{U}$ is acyclic for $\mathcal{F}$ if $\forall\, U_0, \ldots, U_n \in \mathcal{U}$ & $\forall\, i \geq 0$, $H^i(U_0 \cap \ldots \cap U_n, \mathcal{F}|_{U_0 \cap \ldots \cap U_n}) = 0$

[Garrett §10]

<u>Theorem</u> (Leray): $\check{H}^n(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^n(X, \mathcal{F})$ if the above holds.

<u>Proof</u>.

<u>For $i = 0$</u>: ✓ $(=\cdot 11)$

<u>For $i > 0$</u>: Use induction.

Embed $\mathcal{F}$ in an injective sheaf $\mathcal{J}$ & let $\mathcal{Q}$ be the quotient sheaf then we have

$$0 \to \mathcal{F} \to \mathcal{J} \to \mathcal{Q} \to 0$$

For all non-empty $n+1$ intersections $U = U_1 \cap \cdots U_{n+1}$ of elements of the cover $\mathcal{U}$, we get a LES of cohomology (as right derived functor)

$$0 \to H^0(U, \mathcal{F}|_U) \to H^0(U, \mathcal{J}|_U) \to H^0(U, \mathcal{Q}|_U) \to H^1(U, \mathcal{F}|_U) \to$$

$$\cdots \to H^i(U, \mathcal{F}|_U) \to H^i(U, \mathcal{J}|_U) \to H^i(U, \mathcal{Q}|_U) \to \cdots$$

By the acyclicity hypothesis of $\mathcal{U}$ for $\mathcal{F}$, $H^1(U, \mathcal{F}|_U) = 0$

so we get (since $H^0(U,-) = \Gamma(X,-)$)

$$0 \to \Gamma(U,\mathcal{F}) \to \Gamma(U,\mathcal{G}) \to \Gamma(U,Q) \to 0$$

And from higher degrees of LES, invoking the acyclicity of $\mathcal{U}$ for $\mathcal{F}$ & acyclicity of injective sheafs, we can conclude that for $i>0$, $\check{H}^i(U,Q) = 0$

i.e the same cover $\mathcal{U}$ is also acyclic for quotient sheaf Q.

- By taking products of the exact sequence for the cover $\mathcal{U}$ an exact sequence of complexes

$$0 \to C^\circ(\mathcal{U},\mathcal{F}) \to C^\circ(\mathcal{U},\mathcal{G}) \to C^\circ(\mathcal{U},Q) \to 0$$

Getting this SES was the point for assuming acyclicity of $\mathcal{U}$ for $\mathcal{F}$.

Now LES of cohomology of this complexes is

$$0 \to \check{H}^0(\mathcal{U},\mathcal{F}) \to \check{H}^0(\mathcal{U},\mathcal{G}) \to \check{H}^0(\mathcal{U},Q) \cdots$$

$$\to \check{H}^1(\mathcal{U},\mathcal{F}) \to \check{H}^1(\mathcal{U},\mathcal{G}) \to \check{H}^1(\mathcal{U},Q) \to \cdots$$

$$\cdots \to \check{H}^i(\mathcal{U},\mathcal{F}) \to \check{H}^i(\mathcal{U},\mathcal{G}) \to \check{H}^i(\mathcal{U},Q) \to \cdots$$

Then $\check{H}^i(\mathcal{U},\mathcal{G}) = 0$ because $\mathcal{G}$ is injective.

So the LES breaks into smaller exact sequences

$$\left.\begin{array}{l} 0 \to \check{H}^0(\mathcal{U},\mathcal{F}) \to \check{H}^0(\mathcal{U},\mathcal{G}) \to \check{H}^0(\mathcal{U},Q) \to \check{H}^1(\mathcal{U},\mathcal{F}) \to 0 \\ 0 \to \check{H}^{i-1}(\mathcal{U},Q) \to \check{H}^i(\mathcal{U},\mathcal{F}) \to 0 \quad \text{for } \forall i \geq 1. \end{array}\right\} \star$$

· Now for ($R^0\Gamma$) sheaf cohomology.

We have analogous exact seq. (*) from the LES from
$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0,$$ (also assuming injectivity like above)

· Then Comparison map:

Natural vertical maps: Commutative due to injective resolutions.

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "id"] & \check{H}^0(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d] & \check{H}^0(\mathcal{U},\mathcal{Q}) \arrow[r] \arrow[d] & \check{H}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(\mathcal{U},\mathcal{F}) \arrow[r] & H^0(\mathcal{U},\mathcal{G}) \arrow[r] & H^0(\mathcal{U},\mathcal{Q}) \arrow[r] & H^1(\mathcal{U},\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

By diagram chasing this implies $\check{H}^1(\mathcal{U},\mathcal{F}) \cong H^1(X,\mathcal{F})$.

Furthermore, we have

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^{i-1}(\mathcal{U},\mathcal{Q}) \arrow[r] \arrow[d] & \check{H}^i(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{i-1}(X,\mathcal{Q}) \arrow[r] & H^i(X,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

with vertical arrows as natural maps.

The cover $\mathcal{U}$ is acyclic for quotient $\mathcal{Q}$ so by "induction" we know $\forall i$, $\check{H}^{i-1}(\mathcal{U},\mathcal{Q}) = 0$. Then by induction all natural vertical maps must be isomorphisms. ∎

Remark. For Paracompact spaces, acyclicity for covers $\mathcal{U}$ hold & the hypothesis of Leray thm is true. (Next lecture)

§. 17. Connection to Picard group

(Classification of line bundles on X can be done w Čech methods)

Let $(X, \mathcal{O}_X)$ be a ringed space.

- Def. An invertible $\mathcal{O}$-mod M is given by following data:

  a. an open covering $\mathcal{U} = (U_i)_{i\in I}$ of X s.t. $\forall i \in I$

$$M|_{\mathcal{U}_i} \cong \mathcal{O}|_{\mathcal{U}_i}$$

  b. for each $i, j \in I$, $\exists$ isom. of $\mathcal{O}|_{(U_i \cap U_j)}$-modules

$$(*) \quad \mathcal{O}|_{(\mathcal{U}_i \cap \mathcal{U}_j)} \xrightarrow{\sim} \mathcal{O}|_{(\mathcal{U}_j \cap \mathcal{U}_i)}$$

$$\left(\text{and also } \cong M|_{(U_i \cap U_j)}\right)$$

Recall: 4.5.2, 3

- $\Gamma(X, \mathcal{O}) \to \mathrm{End}\, \mathcal{O} \left(= \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O})\right)$ is ring isom w/ composition as multiplication, sending $s \in \Gamma(X, \mathcal{O})$ to an endomorphism given over $\mathcal{U} \overset{\text{open}}{\subset} X$ by multiplication by restriction $\rho^X_{\mathcal{U}}(s)$:

$$\Gamma(\mathcal{U}, \mathcal{O}) \to \Gamma(\mathcal{U}, \mathcal{O}) : t \mapsto \rho^X_{\mathcal{U}}(s) \cdot t$$

& hence, $\mathrm{Aut}\, \mathcal{O} \xrightarrow{\sim} \Gamma(X, \mathcal{O})^*$ the gp. of units of $\Gamma(X, \mathcal{O})$

This implies (b) is equivalent to giving a unit $f_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O})$ $\forall (i,j) \in I \times I$

(so that isom. of (*) is multiply by f_{ij})

- An assignment $U \mapsto (\Gamma(U, \mathcal{O}))^* =$ gp. of units of $\Gamma(U, \mathcal{O})$ defines a sheaf $\mathcal{O}^*$ of ab. gps (under multiplication) & the f_{ij} gives an element

$$f = (f_{ij})_{(i,j) \in I_1} \in C^1(\mathcal{U}, \mathcal{O}^*)$$

- Note, Isoms. of (b) (*) are compatible w triple intersection $U_i \cap U_j \cap U_k$, so f is in fact a cocycle, i.e.

$$f \in \ker(d_1 : C^1(\mathcal{U}, \mathcal{O}^*) \to C^2(\mathcal{U}, \mathcal{O}^*))$$
$$=: Z^1(\mathcal{U}, \mathcal{O}^*)$$

- Conversely / Also note

given $f \in Z^1(\mathcal{U}, \mathcal{O}^*)$ can construct $\mathcal{O}$-mod M by glueing copies of $\mathcal{O}|_{(U_i \cap U_j)}$ (by f as in (b))

- So we have defined a map

$$\xi : Z^1(\mathcal{U}, \mathcal{O}^*) \to \mathrm{Pic}(X)$$

$\mathrm{im}\ \xi := \{$ isom. classes of invertible sheaves trivialized by $\mathcal{U}\}$

i.e $\forall i \in I$ $\quad M|_{U_i} \cong \mathcal{O}|_{U_i}$

- ξ is morph. of ab. gps by

Composition of cocycles $\longmapsto$ operation on $\mathrm{Pic}\ X$ by ⊗

- Note: If $f_{ij} \in \ker \xi$ then the invertible sheaf M constructed from f as above is trivial.

i.e. $M \cong \mathcal{O}$

- Now we have a global section $\Gamma(X, \mathcal{O}^*) \ni 1$,

let $g_i \in \Gamma(U_i, M)$ be the corresponding section.

Then (by b) i.e. multiply w f_{ij}, we get

$\forall i, j \in I \quad g_j = f_{ij} \circ g_i$ on $U_i \cap U_j$

So f is a coboundary i.e.

$$f \in \operatorname{Im}\left(d_0 : C^0(\mathcal{U}, \mathcal{O}^*) \to C^1(\mathcal{U}, \mathcal{O}^*)\right)$$

- So ξ induces an injection with same im (ξ)

$$H^1(\mathcal{U}, \mathcal{O}^*) \to \operatorname{Pic} X$$

(Since every inv. sheaf is trivial over some covering & refinement maps are compatible w ξ), we have

- Theorem. $\exists$ Isom. of Ab gps

$$H^1(X, \mathcal{O}^*) \cong \check{H}^1(X, \mathcal{O}^*) \cong \operatorname{Pic} X.$$

- Example. $(X, \mathcal{O})$ complex mfd (continuous & analytic)

Then $\exists$ sheaf morph $\mathcal{O} \to \mathcal{O}^*$

$\mathbb{C}$ valued $f \longmapsto e^{2\pi i f}$

& a SES: $0 \to \mathbb{Z} \to \mathcal{O} \longrightarrow \mathcal{O}^* \to 0$

Const sheaf (of $\mathbb{Z}$-valued function)

& associated map $\operatorname{Pic} X = H^1(X, \mathcal{O}^*) \xrightarrow{\partial} H^2(X, \mathbb{Z})$.

§. References :

- Tennison : Sheaf Theory
- Grothendieck : English translation of "Sur quelques points d'algèbre homologique"
- Tamme : Introduction to Étale Cohomology
- Garrett : Notes Sheaf Cohomology